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## JIAA TR - 29

# ON SOURCE RADIATION

Harold Levine



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The power output from given sources is usually ascertained via an energy flux integral over the normal directions to a remote (far field) surface; an alternative procedure, which utilizes an integral that specifies the direct rate of working by the source on the resultant field, is described and illustrated for both point and continuous source distributions. A comparison between the respective procedures is made in the analysis of sound radiated from a periodic dipole source whose axis performs a periodic plane angular movement about a fixed direction. Thus, adopting a conventional approach, Sretenskii (1956) characterizes the rotating dipole in terms of an infinite number of stationary ones along a pair of orthogonal directions in the plane and, through the far field representation of the latter, arrives at a series development for the instantaneous radiated power, whereas the local manner of power calculation dispenses with the equivalent infinite aggregate of sources and yields a compact analytical result.

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# ON SOURCE RADIATION

by

Harold Levine

Joint Institute for Aeronautics and Acoustics

Stanford University

## §1. Introduction

Determinations of the energy radiated by different source types (electromagnetic, acoustic, or elastic) are generally based on the far field or asymptotic form of the wave function, although there exists an independent way, less well exploited, of calculating the radiation from a system: this involves a direct consideration of the rate at which the source (electrical current, acoustical force, ...) does work on the field and thus supplies the energy loss by radiation.

Having regard, specifically, for the microscopic (Maxwell-Lorentz) equations

$$\nabla \times \vec{b} = \frac{1}{c}\frac{\partial \vec{e}}{\partial t} + \frac{1}{c}\vec{j} \quad , \qquad \nabla \cdot \vec{e} = \rho$$
$$\nabla \times \vec{e} = -\frac{1}{c}\frac{\partial \vec{b}}{\partial t} \, , \qquad \nabla \cdot \vec{b} = 0 \tag{1}$$

of an electromagnetic field described by the two vectors $\vec{e}(\vec{r},t)$ (the electric field intensity) and $\vec{b}(\vec{r},t)$ (the magnetic field intensity), with the given source functions $\vec{j}(\vec{r},t)$ (the current density) and $\rho(\vec{r},t)$ (the charge density), it is readily shown that

$$\frac{\partial W}{\partial t} + \nabla \cdot \vec{S} = -\vec{j} \cdot \vec{e} \tag{2}$$

where

$$W = \frac{e^2 + b^2}{2} \tag{3}$$

represents the electromagnetic energy density and

$$\vec{S} = c\vec{e} \times \vec{b} \tag{4}$$

characterizes the energy flux (Poynting) vector. An integrated version of the local energy balance relation (2),

$$-\int \vec{j} \cdot \vec{e}\, d\vec{r} = \frac{d\varepsilon}{dt} + P \ , \tag{5}$$

associates the rate of working by the source current with the combined temporal change of the total electromagnetic energy of the system,

$$\varepsilon = \int \frac{e^2 + b^2}{2}\, d\vec{r} \ , \tag{6}$$

and the net amount of energy radiation therefrom per unit time, or power output

$$P = \int \vec{n} \cdot c\vec{e} \times \vec{b}\, dA \ , \tag{7}$$

reckoned on a distant surface with outward (unit) normal vector $\vec{n}$ . Thus, if the (activity) integral be displayed in the component form (5), both the stored and radiated energy measures are independently and simultaneously obtainable.

Evidently, the electric field within the source region alone has a relevance to the aforesaid integral and if the spatial extent of this region is sufficiently limited so that the charge/current density distributions undergo small change in the time required for light to traverse the region at speed $c$ , the effect of retardation is slight; then the source functions

with a retarded time argument which enter into expressions for the field intensities can be developed in powers of $1/c$, viz.,

$$\rho(\vec{r}, t - \frac{|\vec{r}-\vec{r}'|}{c}) = \rho(\vec{r},t) - \frac{1}{c}|\vec{r}-\vec{r}'| \frac{\partial}{\partial t} \rho(\vec{r},t) + \frac{1}{2c^2}|\vec{r}-\vec{r}'|^2 \frac{\partial^2}{\partial t^2} \rho(\vec{r},t) - \frac{1}{6c^3} |\vec{r}-\vec{r}'|^3 \frac{\partial^3}{\partial t^3} \rho(\vec{r},t) + \dots . \tag{8}$$

The leading order estimate for the power radiation deduced from the integral in (5) on the basis of such a development proves to be

$$P = \frac{2}{3c^3} \frac{1}{4\pi} \left(\frac{d^2 p}{dt^2}\right)^2 \tag{9}$$

where $\vec{p}(t) = \int \vec{r} \rho(\vec{r},t) d\vec{r}$ denotes the electric dipole moment of the source. In the particular circumstance of a dipole moment which oscillates harmonically at a single frequency, say

$$\vec{p}(t) = \vec{p}_0 \cos \omega t , \tag{10}$$

the rate of emission of energy has the instantaneous magnitude

$$P = \frac{2}{3c^3} \frac{\omega^4}{4\pi} p_0^2 \cos^2 \omega t \tag{11}$$

and fluctuates about an average value

$$\overline{P} = \frac{\omega^4}{12\pi c^3} p_0^2 . \tag{12}$$

The characterization (9) also obtains after integrating the energy flux over a large spherical surface (in the far field), whose radius merely determines the time of emission of the observed field; this implies, if the surface radius $r$ is centered on a harmonically varying electric dipole, that

$$P = \frac{2}{3c^3} \frac{\omega^4}{4\pi} p_0^2 \cos^2 \omega(t - \frac{r}{c}) ,$$

with the same average value as in (12).

Having regard, in a similar context, for the linearized system of equations which link the pressure and velocity disturbances $p(\vec{r},t)$ and $\vec{v}(\vec{r},t)$, in a homogeneous medium with density $\rho_0$ and sound speed $c$, to the action of a prescribed force $\vec{F}(\vec{r},t)$, viz.,

$$\frac{1}{c^2}\frac{\partial p}{\partial t} + \rho_0 \nabla \cdot \vec{v} = 0$$

and (13)

$$\rho_0 \frac{\partial \vec{v}}{\partial t} = -\nabla p + \vec{F} ,$$

it is readily verified that the local energy balance relation takes the form

$$\frac{\partial W}{\partial t} + \nabla \cdot \vec{S} = \vec{F} \cdot \vec{v} ; \tag{14}$$

here

$$W = \frac{1}{2}\rho_0 v^2 + \frac{1}{2}\frac{p^2}{\rho_0 c^2} \tag{15}$$

designates the energy density and

$$\vec{S} = p\vec{v} \tag{16}$$

is the acoustic energy flux vector. The energy radiation rate from a region wherein an external force is applied can thus be inferred from the volume integral of $\vec{F} \cdot \vec{v}$, after isolating a time derivative contribution thereto.

If the motions are irrotational and initiated by a source term $Q(\vec{r},t)$ on the right-hand side of the first equation (13), then

$$\vec{v} = -\nabla\phi , \tag{17}$$

and

$$p = \rho_0 \frac{\partial \phi}{\partial t} \tag{18}$$

(where the latter representation follows from the second equation (13)) and the velocity potential $\phi(\vec{r},t)$ satisfies an inhomogeneous wave equation

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{1}{\rho_0} Q(\vec{r},t) \ . \tag{19}$$

Multiplication in (19) by $\partial\phi/\partial t$ and subsequent rearrangement yields the balance equation

$$\frac{\partial W}{\partial t} + \nabla \cdot \vec{S} = Q(\vec{r},t)\frac{\partial\phi}{\partial t} \tag{20}$$

with energy density and flux measures,

$$W = \frac{1}{2}\rho_0(\nabla\phi)^2 + \frac{1}{2\rho_0 c^2}\left(\frac{\partial\phi}{\partial t}\right)^2 \tag{21}$$

and

$$\vec{S} = -\rho_0\frac{\partial\phi}{\partial t}\nabla\phi \tag{22}$$

that are the appropriate versions of (15), (16) when (17), (18) hold. Thus, the product $Q(\vec{r},t)\frac{\partial\phi}{\partial t}$ serves as a measure of the source activity and its volume integral can be utilized to determine the net energy radiation.

The actual manner of calculating energy radiation by analysis of integrals over a source region, which this paper aims to describe, commences in the next section. Point source models are adopted at first, thereby allowing comparison with results otherwise found and, in particular, those obtained by Sretenskii (1956) for rotating dipoles. A velocity potential and the scalar wave equation underlie the initial presentation, whereas the local velocity and a vector wave equation take over the corresponding roles in the later account of technique. Finally, some consideration is given to extended source distributions.

§2. Sound Radiation by Simple and Dipole Sources

On employing the retarded potential solution of the inhomogeneous equation (19),

$$\phi(\vec{r},t) = \frac{1}{4\pi\rho_0} \int \frac{Q(\vec{r}\,',t - \frac{|\vec{r}-\vec{r}\,'|}{c})}{|\vec{r}-\vec{r}\,'|} d\vec{r}\,' , \tag{23}$$

and assuming that the source function $Q(\vec{r},t)$ has a localized nature, i.e.,

$$Q(\vec{r},t) = \delta(\vec{r})f(t) , \tag{24}$$

there emerges the spherical wave function

$$\phi(r,t) = \frac{1}{4\pi\rho_0} \frac{f(t-\frac{r}{c})}{r} \tag{25}$$

descriptive of a simple source, with variable strength $f(t)$, at the origin. When the latter and its time derivative are expanded in powers of $1/c$, it follows that

$$\phi(r,t) = \frac{1}{4\pi\rho_0} \left\{ \frac{f(t)}{r} - \frac{1}{c}\dot{f}(t) + \frac{r}{2c^2}\ddot{f}(t) + O(\frac{r^2}{c^3}) \right\} \tag{26}$$

and

$$\frac{\partial\phi(r,t)}{\partial t} = \frac{1}{4\pi\rho_0} \left\{ \frac{\dot{f}(t)}{r} - \frac{1}{c}\ddot{f}(t) + O(\frac{r}{c^2}) \right\} \tag{27}$$

with dots symbolizing differentiation. Substitution of (27) into the source activity integral [cf. (20)] yields

$$I = \int Q(\vec{r},t) \frac{\partial\phi}{\partial t} d\vec{r} = \int \delta(\vec{r})f(t) \frac{\partial\phi}{\partial t} d\vec{r}$$

$$= \int \delta(\vec{r})f(t) \left\{ \frac{1}{4\pi\rho_0}\frac{\dot{f}(t)}{r} - \frac{1}{4\pi\rho_0 c}\ddot{f}(t) + O(\frac{r}{c^2}) \right\} d\vec{r}$$

$$= \frac{d}{dt}\left\{\frac{f^2(t)}{2}\right\} \cdot \frac{1}{4\pi\rho_0}\int \frac{\delta(\vec{r})}{r} d\vec{r} - \frac{1}{4\pi\rho_0 c} f(t)\ddot{f}(t) , \tag{28}$$

and, inasmuch as

$$f(t)\ddot{f}(t) = \frac{d}{dt}\,(f(t)\dot{f}(t)) - (\dot{f}(t))^2 \;,$$

it can be deduced that.

$$P = \frac{1}{4\pi\rho_0 c}\,(\dot{f}(t))^2 > 0 \tag{29}$$

expresses the positive definite rate of energy radiation. If the source strength varies harmonically, say

$$f(t) = m \cos \omega t \;,$$

then

$$P = \frac{\omega^2 m^2}{4\pi\rho_0 c} \sin^2 \omega t \tag{30}$$

with an average value

$$\overline{P} = \frac{\omega^2 m^2}{8\pi\rho_0 c} \;. \tag{31}$$

The fact that the time derivative term in (28) contains a divergent integral, $\int \frac{\delta(\vec{r})}{r}\,d\vec{r}$ , is attributable to the idealization of a point source.

Let the point source function (24) be replaced by another,

$$Q(\vec{r},t) = \frac{\partial}{\partial x}\,\delta(\vec{r})f(t) \;, \tag{32}$$

which befits a dipole oriented in the x-direction; then

$$\begin{aligned}
\phi(\vec{r},t) &= \frac{1}{4\pi\rho_0}\frac{\partial}{\partial x}\left(\frac{f(t-\frac{r}{c})}{r}\right) \\
&= \frac{1}{4\pi\rho_0}\frac{\partial}{\partial x}\left\{\frac{f(t)}{r} - \frac{1}{c}\dot{f}(t) + \frac{r}{2c^2}\ddot{f}(t) - \frac{r^2}{6c^3}\dddot{f}(t) + 0(\frac{r^3}{c^4})\right\} \\
&= \frac{1}{4\pi\rho_0}\left\{f(t)\frac{\partial}{\partial x}(\frac{1}{r}) + \frac{x}{2rc^2}\ddot{f}(t) - \frac{x}{3c^3}\dddot{f}(t) + 0(\frac{xr}{c^4})\right\}
\end{aligned} \tag{33}$$

and

$$\frac{\partial\phi(\vec{r},t)}{\partial t} = \frac{1}{4\pi\rho_0}\left\{\dot{f}(t)\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \frac{x}{2rc^2}\dddot{f}(t) - \frac{x}{3c^3}\ddddot{f}(t) + 0\left(\frac{xr}{c^4}\right)\right\}$$

so that

$$\begin{aligned} I &= \int Q(\vec{r},t)\frac{\partial\phi}{\partial t}\,d\vec{r} = \int \frac{\partial}{\partial x}\delta(\vec{r})f(t)\frac{\partial\phi}{\partial t}\,d\vec{r} \\ &= \frac{d}{dt}\left\{\frac{f^2(t)}{8\pi\rho_0}\int\delta(\vec{r})\frac{\partial^2}{\partial x^2}\left(-\frac{1}{r}\right)d\vec{r}\right\} + \frac{1}{2c^2}f(t)\dddot{f}(t)\int\delta(\vec{r})\frac{\partial}{\partial x}\left(\frac{x}{r}\right)d\vec{r} \\ &\quad + \frac{1}{12\pi\rho_0 c^3}f(t)\ddddot{f}(t)\ . \end{aligned} \tag{34}$$

After invoking the determination

$$\begin{aligned} \int\delta(\vec{r})\frac{\partial}{\partial x}\left(\frac{x}{r}\right)d\vec{r} &= \int\delta(x)\delta(y)\delta(z)\frac{y^2+z^2}{r^3}\,dx\,dy\,dz \\ &= \lim_{y,z\to 0;\,x\to 0}\left(\frac{y^2+z^2}{r^3}\right) = 0 \end{aligned}$$

and the rearrangement

$$f(t)\ddddot{f}(t) = \frac{d}{dt}\left[f(t)\dddot{f}(t) - \dot{f}(t)\ddot{f}(t)\right] + (\ddot{f}(t))^2$$

the power output of the dipole is identified as

$$P = \frac{1}{12\pi\rho_0 c^3}(\ddot{f}(t))^2\ . \tag{35}$$

Thus, if the dipole moment varies harmonically, and $f = m\cos\omega t$ , the expressions

$$P = \frac{\omega^4 m^2}{12\pi\rho_0 c^3}\cos^2\omega t \tag{36}$$

and

$$\overline{P} = \frac{\omega^4 m^2}{24\pi\rho_0 c^3} \tag{37}$$

obtain.

Consider next a dipole with harmonically varying moment, whose axis rotates at a uniform angular velocity in the x,y plane; the affiliated wave function is

$$\phi(\vec{r},t) = \frac{1}{4\pi\rho_0}\left\{\frac{\partial}{\partial x}\left(\frac{f_1(t-\frac{r}{c})}{r}\right) + \frac{\partial}{\partial y}\left(\frac{f_2(t-\frac{r}{c})}{r}\right)\right\} \tag{38}$$

with

$$f_1(t) = m\cos\Omega t\cos\omega t = \frac{1}{2}m\{\cos(\omega+\Omega)t + \cos(\omega-\Omega)t\}$$

and (39)

$$f_2(t) = m\sin\Omega t\cos\omega t = \frac{1}{2}m\{\sin(\omega+\Omega)t - \sin(\omega-\Omega)t\}$$

where $\omega$, $\Omega$ designate the natural and precession frequencies, respectively. On utilizing the expansions for $f_1$, $f_2$ in (38) which feature rising powers of $r/c$ and the concomitant source function

$$Q(\vec{r},t) = \{f_1(t)\frac{\partial}{\partial x} + f_2(t)\frac{\partial}{\partial y}\}\delta(\vec{r}) \tag{40}$$

it can be verified that

$$I = \int Q(\vec{r},t)\frac{\partial\phi}{\partial t}d\vec{r} = \frac{d\varepsilon}{dt} + P$$

with

$$\begin{aligned}\varepsilon = \frac{1}{8\pi\rho_0}&\left\{f_1^2(t)\int\delta(\vec{r})\frac{\partial^2}{\partial x^2}(-\frac{1}{r})d\vec{r} + f_2^2(t)\int\delta(\vec{r})\frac{\partial^2}{\partial y^2}(-\frac{1}{r})d\vec{r}\right\}\\ &+\frac{1}{4\pi\rho_0}f_1(t)f_2(t)\int\delta(\vec{r})\frac{\partial^2}{\partial x\partial y}(-\frac{1}{r})d\vec{r} + \frac{1}{12\pi\rho_0c^3}\left\{f_1(t)\dddot{f}_1(t) - \dot{f}_1(t)\ddot{f}_1(t)\right.\\ &\left.+ f_2(t)\dddot{f}_2(t) - \dot{f}_2(t)\ddot{f}_2(t)\right\}\end{aligned}$$

and

$$P = \frac{1}{12\pi\rho_0c^3}\{(\ddot{f}_1(t))^2 + (\ddot{f}_2(t))^2\}\ . \tag{41}$$

Since

$$\ddot{f}_1(t) = -\frac{m}{2}\left\{(\omega+\Omega)^2 \cos(\omega+\Omega)t + (\omega-\Omega)^2 \cos(\omega-\Omega)t\right\}$$

and

$$\ddot{f}_2(t) = -\frac{m}{2}\left\{(\omega+\Omega)^2 \sin(\omega+\Omega)t - (\omega-\Omega)^2 \sin(\omega-\Omega)t\right\}$$

the power radiated by the rotating dipole has the instantaneous and time average magnitudes,

$$P = \frac{1}{12\pi\rho_0 c^3}\left(\frac{m^2}{4}\right)\left\{(\omega+\Omega)^4 + (\omega-\Omega)^4 + 2(\omega^2-\Omega^2)^2 \cos 2\omega t\right\}$$

$$= \frac{m^2}{24\pi\rho_0 c^3}\left\{\omega^4 + 6\omega^2\Omega^2 + \Omega^4 + (\omega^2-\Omega^2)^2 \cos 2\omega t\right\} \tag{42}$$

and

$$\overline{P} = \frac{m^2}{24\pi\rho_0 c^3}\left\{\omega^4 + 6\omega^2\Omega^2 + \Omega^4\right\} , \tag{43}$$

respectively, the latter in agreement with the prediction of Sretenskii.

If the axis of the dipole oscillates symmetrically, within a limited angular sector, about the x-direction, the characteristic source functions in (38) are expressed by

$$f_1(t) = m \cos \omega t \cos(\gamma \cos \Omega t)$$

and (44)

$$f_2(t) = m \cos \omega t \sin(\gamma \cos \Omega t) .$$

Straightforward differentiation of these functions , as called for in the power representation (41), yields the explicit result

$$P = \frac{m^2}{12\pi\rho_0 c^3}\left\{\cos^2 \omega t(\omega^2+\gamma^2\Omega^2 \sin^2 \Omega t)^2 + (2\gamma\omega\Omega \sin \omega t \sin \Omega t - \gamma\Omega^2 \cos \omega t \cos \Omega t)^2\right\}$$

$$= \frac{m^2}{12\pi\rho_0 c^3}\Big\{\cos^2 \omega t(\omega^4 + 2\gamma^2\omega^2\Omega^2 \sin^2 \Omega t + \gamma^2\Omega^4 \cos^2 \Omega t + \gamma^4\Omega^4 \sin^4 \Omega t)$$
$$+ 4\gamma^2\omega^2\Omega^2 \sin^2 \omega t \sin^2 \Omega t - \gamma^2\omega\Omega^3 \sin 2\omega t \sin 2\Omega t\Big\} \qquad (45)$$

wherein the individual magnitudes of $\omega$ and $\Omega$ can be arbitrarily set.

Sretenskii, whose power analysis utilizes both the far field measure of the dipole source and the development

$$e^{i\gamma \cos \Omega t} = \sum_{n=-\infty}^{\infty} i^n J_n(\gamma) e^{in\Omega t} \qquad (46)$$

(containing integer order Bessel functions $J_n$ ), arrives at the result

$$P = \frac{m^2}{12\pi\rho_0 c^3} [\Gamma_1^2(\tau) + \Gamma_2^2(\tau)] \quad , \quad \tau = t - \frac{r}{c} \qquad (47)$$

where

$$\Gamma_1(\tau) + i\Gamma_2(\tau) = \sum_{n=-\infty}^{\infty} (\omega+n\Omega)^2 e^{\frac{1}{2} in\pi} J_n(\gamma) \cos(\omega+n\Omega)\tau \; . \qquad (48)$$

He defines, furthermore, an average value of $P$ , viz.

$$\overline{P} = \lim_{T\to\infty} \left\{\frac{1}{T} \int_0^T P(t)\, dt\right\}, \qquad (49)$$

and, subject to the proviso that no relationship of the form $2\omega/\Omega = N$ (integral) exists, obtains the expression

$$\overline{P} = \frac{m^2}{24\pi\rho_0 c^3} \sum_{n=-\infty}^{\infty} (\omega+n\Omega)^4 J_n^2(\gamma) \qquad (50)$$

which, on the basis of Bessel summation, acquires its ultimate version

$$\overline{P} = \frac{m^2}{24\pi\rho_0 c^3} \left\{\omega^4 + 3\omega^2\Omega^2\gamma^2 + \Omega^4\gamma^2(\tfrac{1}{2}+\tfrac{3}{8}\gamma^2)\right\}. \qquad (51)$$

The power representation (45), which has finitely many terms in contrast with (47) - (48), lends itself to integration and subsequent averaging over any span of time, say $0 < t < T$ ; specifically,

$$\int_0^T P\,dt = \frac{m^2}{24\pi\rho_0 c^3}\Big\{T(\omega^4 + 3\gamma^2\omega^2\Omega^2 + \gamma^2\Omega^4(\tfrac{1}{2}+\tfrac{3}{8}\gamma^2))$$

$$+ (\frac{\omega^3}{2} - \frac{1}{2}\gamma^2\omega\Omega^2 + \frac{1}{4}\gamma^2\frac{\Omega^4}{\omega} + \frac{3}{16}\gamma^4\frac{\Omega^4}{\omega})\sin 2\omega T$$

$$+ (-\frac{3}{2}\gamma^2\omega^2\Omega + \frac{1}{4}\gamma^2\Omega^3 - \frac{3}{16}\gamma^4\Omega^3)\sin 2\Omega T$$

$$+ (\frac{1}{4}\gamma^2\omega^2\Omega^2 + \frac{1}{2}\gamma^2\omega\Omega^3 + \frac{1}{8}\gamma^2\Omega^4 - \frac{5}{16}\gamma^4\Omega^4)\frac{\sin 2(\omega+\Omega)T}{\omega+\Omega}$$

$$+ (\frac{1}{4}\gamma^2\omega^2\Omega^2 - \frac{1}{2}\gamma^2\omega\Omega^3 + \frac{1}{8}\gamma^2\Omega^4 - \frac{5}{16}\gamma^4\Omega^4)\frac{\sin 2(\omega-\Omega)T}{\omega-\Omega}$$

$$+ \frac{\gamma^4\Omega^4}{32}\left(\frac{\sin 2(\omega+2\Omega)T}{\omega+2\Omega} + \frac{\sin 2(\omega-2\Omega)T}{\omega-2\Omega}\right)\Big\}\ . \qquad (52)$$

Thus, the non-trigonometric terms of (52) dominate in the limit $T \to \infty$ and the average value $\overline{P}$, as defined by (49), corresponds exactly with (51). If the span

$$T = 2\pi/\Omega$$

equals the precessional period of the dipole, the appertaining average power

$$\frac{\Omega}{2\pi}\int_0^{2\pi/\Omega} P\,dt = \overline{P}$$

$$= \frac{m^2}{24\pi\rho_0 c^3}\Big\{\omega^4 + 3\gamma^2\omega^2\Omega^2 + \gamma^2\Omega^4(\tfrac{1}{2}+\tfrac{3}{8}\gamma^2)$$

$$+ \frac{\Omega}{2\pi}(\frac{1}{2}\omega^3 - \frac{1}{2}\gamma^2\omega\Omega^2 + \frac{1}{4}\gamma^2\frac{\Omega^4}{\omega} + \frac{3}{16}\gamma^4\frac{\Omega^4}{\omega})\sin\frac{4\pi\omega}{\Omega}$$

$$+ \Bigg[\frac{\frac{1}{4}\gamma^2\omega^2\Omega^2 + \frac{1}{2}\gamma^2\omega\Omega^3 + \frac{1}{8}\gamma^2\Omega^4 - \frac{5}{16}\gamma^4\Omega^4}{\omega+\Omega} \qquad (53)$$

$$+ \frac{\frac{1}{4}\gamma^2\omega^2\Omega^2 - \frac{1}{2}\gamma^2\omega\Omega^3 + \frac{1}{8}\gamma^2\Omega^4 - \frac{5}{16}\gamma^4\Omega^4}{\omega-\Omega}\Bigg]\frac{\Omega}{2\pi}\sin\frac{4\pi\omega}{\Omega}$$

$$+ \frac{\gamma^4\Omega^4}{32}\left(\frac{1}{\omega+2\Omega} + \frac{1}{\omega-2\Omega}\right)\frac{\Omega}{2\pi}\sin\frac{4\pi\omega}{\Omega}\Big\}$$

has the limiting estimate

$$\overline{P} = \frac{m^2}{24\pi\rho_0 c^3}\left\{\gamma^2\Omega^4(1 + \frac{3}{4}\gamma^2) + 0(\Omega^2\omega^2)\right\} \quad , \quad \Omega >> \omega \tag{54}$$

where the explicit (leading) terms are multiplied by a factor of 2 relative to the like pair inferred from (51).

The average power relative to a period based on the intrinsic frequency of the dipole, $T = 2\pi/\omega$ , can be directly ascertained from (51) and its estimate readily secured if $\omega >> \Omega$ .

## §3. Sound Radiation from Point Forces

To commence the analysis of power radiation associated with a vectorial source function, namely an impressed force, suppose that the latter has a variable magnitude and a fixed orientation, i.e.,

$$\vec{F}(\vec{r},t) = \vec{i}\,\mathscr{F}(\vec{r},t) \tag{55}$$

where $\vec{i},\vec{j},\vec{k}$ are the trio of unit vectors in the x,y,z directions, respectively. Then

$$\nabla \times \vec{F} = \vec{j}\,\frac{\partial\mathscr{F}}{\partial z} - \vec{k}\,\frac{\partial\mathscr{F}}{\partial y} \tag{56}$$

and the vorticity relation which is an immediate consequence of the second of the basic equations (13), namely

$$\rho_0 \frac{\partial}{\partial t} \nabla \times \vec{v} = \nabla \times \vec{F} , \tag{57}$$

admits the integral

$$\nabla \times \vec{v} = \frac{1}{\rho_0}\left\{\vec{j} \int_{-\infty}^{t} \frac{\partial\mathscr{F}}{\partial z}\,dt - \vec{k} \int_{-\infty}^{t} \frac{\partial\mathscr{F}}{\partial y}\,dt\right\} . \tag{58}$$

After eliminating the pressure p from the system (13), it follows that

$$\rho_0 \nabla\nabla \cdot \vec{v} = -\frac{1}{c^2}\frac{\partial}{\partial t}\left[\vec{F} - \rho_0 \frac{\partial \vec{v}}{\partial t}\right]$$

and thus, relying on the vector identity

$$\nabla \times \nabla \times \vec{v} = \nabla\nabla \cdot \vec{v} - \nabla^2\vec{v} \tag{59}$$

together with (58), an inhomogeneous wave equation for $\vec{v}$ obtains

$$\begin{aligned}\nabla^2\vec{v} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{v} &= -\frac{1}{\rho_0 c^2}\frac{\partial \vec{F}}{\partial t} - \frac{1}{\rho_0}\nabla \times \left\{\vec{j}\int_{-\infty}^{t}\frac{\partial \mathscr{F}}{\partial z}dt - \vec{k}\int_{-\infty}^{t}\frac{\partial \mathscr{F}}{\partial y}dt\right\} \\ &= \vec{i}\left\{-\frac{1}{\rho_0 c^2}\frac{\partial \mathscr{F}}{\partial t} + \frac{1}{\rho_0}\int_{-\infty}^{t}\left(\frac{\partial^2 \mathscr{F}}{\partial y^2} + \frac{\partial^2 \mathscr{F}}{\partial z^2}\right)dt\right\} \\ &\quad + \vec{j}\left\{-\frac{1}{\rho_0}\int_{-\infty}^{t}\frac{\partial^2 \mathscr{F}}{\partial x \partial y}dt\right\} + \vec{k}\left\{-\frac{1}{\rho_0}\int_{-\infty}^{t}\frac{\partial^2 \mathscr{F}}{\partial x \partial z}dt\right\} .\end{aligned} \tag{60}$$

Only the x-component of this equation needs to be reckoned with in analysing the local source activity or scalar product of $\vec{F}$ and $\vec{v}$, viz.

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)v_x = -\frac{1}{\rho_0 c^2}\frac{\partial \mathscr{F}}{\partial t} + \frac{1}{\rho_0}\int_{-\infty}^{t}\left(\frac{\partial^2 \mathscr{F}}{\partial y^2} + \frac{\partial^2 \mathscr{F}}{\partial z^2}\right)dt . \tag{61}$$

If the source function $\mathscr{F}(\vec{r},t)$ has a concentrated spatial nature and is temporarily written in the form

$$\mathscr{F}(\vec{r},t) = \delta(\vec{r})\frac{dp}{dt}$$

the subsequent version of (61),

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)v_x = -\frac{1}{\rho_0 c^2}\delta(\vec{r})\ddot{p}(t) + \frac{1}{\rho_0}p(t)\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\delta(\vec{r}) ,$$

admits the retarded time solution

$$v_x(\vec{r},t) = \frac{1}{4\pi\rho_0 c^2}\left\{\frac{\ddot{p}(t-\frac{r}{c})}{r} - c^2\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\frac{p(t-\frac{r}{c})}{r}\right\}$$

with the development

$$v_x(\vec{r},t) = \frac{1}{4\pi\rho_0 c^2}\left\{\frac{\ddot{p}(t)}{r} - \frac{1}{c}\dddot{p}(t) - c^2 p(t)\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\frac{1}{r}\right.$$

$$\left. - \frac{1}{2}\ddot{p}(t)\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right) r + \frac{2}{3c}\dddot{p}(t) + 0(\frac{1}{c^2})\right\} .$$

After identifying and grouping all contributions to the integral

$$I = \int \delta(\vec{r})\dot{p}(t)v_x(\vec{r},t)\, d\vec{r}$$

which are expressible in the form of a time derivative the remainder, viz.

$$P = \frac{1}{4\pi\rho_0 c^3}\cdot\frac{1}{3}(\ddot{p})^2 ,$$

can be interpreted as a measure of power radiation; and, if the replacement $\dot{p}(t) = f(t)$ be made therein, the outcome

$$P = \frac{1}{12\pi\rho_0 c^3}(\dot{f}(t))^2 \tag{62}$$

is suited to the source function

$$\mathscr{F}(\vec{r},t) = \delta(\vec{r})f(t) \tag{63}$$

with a time varying strength $f(t)$ .

The pair of functions

$$f_x(t) = f_0 \cos \Omega t \cos \omega t$$

and (64)

$$f_y(t) = f_0 \sin \Omega t \cos \omega t$$

specify instantaneous components of a force with proper frequency $\omega$ that rotates in the x,y plane at the angular frequency $\Omega$ about its point of application. According to the evident counterpart of (62),

$$P = \frac{1}{12\pi\rho_0 c^3}\left\{(\dot{f}_x(t))^2 + (\dot{f}_y(t))^2\right\} \tag{65}$$

it follows that

$$P = \frac{f_0^2}{24\pi\rho_0 c^3}\left\{\omega^2 + \Omega^2 - (\omega^2-\Omega^2)\cos 2\,\omega t\right\} \tag{66}$$

and the average radiated power,

$$\overline{P} = \frac{f_0^2}{24\pi\rho_0 c^3}(\omega^2+\Omega^2)\ , \tag{67}$$

corresponds precisely with an integral of the far field energy flux (whose expression can be found in Morfey and Tanna (1971)).

A time harmonic point force whose line of action oscillates through a plane angular sector is characterized by the components

$$f_x = f_0 \cos \omega t \cos(\gamma \cos \Omega t)$$

and (68)

$$f_y = f_0 \cos \omega t \sin(\gamma \cos \Omega t)\ .$$

On substituting their first derivatives into (65) the radiated power is found to have the instantaneous magnitude

$$P = \frac{f_0^2}{12\pi\rho_0 c^3}\left\{\omega^2 \sin^2 \omega t + \gamma^2\Omega^2 \cos^2 \omega t \sin^2 \Omega t\right\} \tag{69}$$

and its net accumulation or integral between $t = 0$ and $t = T$ has the explicit form

$$\int_0^T P\,dt = \left(\frac{\omega^2}{2} + \frac{\gamma^2\Omega^2}{4}\right)T - \frac{\omega}{4}\sin 2\omega T + \frac{\gamma^2\Omega^2}{8}\left(\frac{\sin 2\omega T}{\omega} - \frac{\sin 2\Omega T}{\Omega}\right)$$

$$- \frac{\gamma^2\Omega^2}{16}\left(\frac{\sin 2(\omega+\Omega)T}{\omega+\Omega} + \frac{\sin 2(\omega-\Omega)T}{\omega-\Omega}\right) . \tag{70}$$

Hence

$$\overline{P} = \lim_{T\to\infty}\left\{\frac{1}{T}\int_0^T P\,dt\right\}$$

$$= \frac{f_0^2}{24\pi\rho_0 c^3}\left\{\omega^2 + \frac{1}{2}\gamma^2\Omega^2\right\} \tag{71}$$

specifies the long time average power output, and

$$\overline{P} = \frac{f_0^2}{12\pi\rho_0 c^3}\left\{\frac{\omega^2}{2} + \frac{\gamma^2\Omega^2}{4} - \frac{\omega\Omega}{8\pi}\sin\frac{4\pi\omega}{\Omega} + \frac{\gamma^2\Omega^3}{16\pi\omega}\sin\frac{4\pi\omega}{\Omega}\right.$$

$$\left. - \frac{\gamma^2\Omega^3}{32\pi}\sin\frac{4\pi\omega}{\Omega}\left(\frac{1}{\omega+\Omega} + \frac{1}{\omega-\Omega}\right)\right\} \tag{72}$$

specifies the average output during a precessional period $T = 2\pi/\Omega$ . If $\Omega >> \omega$ it follows from (72) that

$$\overline{P} = \frac{f_0^2}{12\pi\rho_0 c^3}\left(\frac{\gamma^2\Omega^2}{2} + 0\left(\frac{\omega^2}{\Omega^2}\right)\right) , \tag{73}$$

and this is nearly double the amount predicted by (71).

## §4. Extended Source Distributions

When the pointwise nature of the source function $Q(\vec{r},t)$ , manifest by the representation (24), is set aside and a series development based on powers of $1/c$ invoked for the appertaining wave function (23), namely

$$\phi(\vec{r},t) = \frac{1}{4\pi\rho_0}\int\left\{\frac{Q(\vec{r}',t)}{|\vec{r}-\vec{r}'|} - \frac{1}{c}\frac{\partial}{\partial t}Q(\vec{r}',t) + \frac{|\vec{r}-\vec{r}'|}{2c^2}\frac{\partial^2}{\partial t^2}Q(\vec{r}',t)\right.$$

$$\left. - \frac{|\vec{r}-\vec{r}'|^2}{6c^3}\frac{\partial^3}{\partial t^3}Q(\vec{r}',t) + \dots\right\} d\vec{r}' , \tag{74}$$

the concomitant version of the source activity integral becomes

$$I = \int Q(\vec{r},t)\,\frac{\partial\phi(\vec{r},t)}{\partial t}\,d\vec{r}$$

$$= \frac{1}{4\pi\rho_0}\int \frac{Q(\vec{r},t)\,\frac{\partial}{\partial t}\,Q(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|}\,d\vec{r}\,d\vec{r}\,' - \frac{1}{4\pi\rho_0 c}\int Q(\vec{r},t)\,\frac{\partial^2}{\partial t^2}\,Q(\vec{r}\,',t)\,d\vec{r}\,d\vec{r}\,'$$

$$+ \frac{1}{8\pi\rho_0 c^2}\int |\vec{r}-\vec{r}\,'|\,Q(\vec{r},t)\,\frac{\partial^3}{\partial t^3}\,Q(\vec{r}\,',t)\,d\vec{r}\,d\vec{r}\,'$$

$$- \frac{1}{24\pi\rho_0 c^3}\int |\vec{r}-\vec{r}\,'|^2 Q(\vec{r},t)\,\frac{\partial^4}{\partial t^4}\,Q(\vec{r}\,',t)\,d\vec{r}\,d\vec{r}\,' + O(1/c^4)\ . \qquad (75)$$

On taking note of the various relations

$$\int \frac{Q(\vec{r},t)\,\frac{\partial}{\partial t}\,Q(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|}\,d\vec{r}\,d\vec{r}\,' = \frac{1}{2}\,\frac{d}{dt}\int \frac{Q(\vec{r},t)Q(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|}\,d\vec{r}\,d\vec{r}\,' \qquad (76)$$

(justified by the symmetric character of the integrand relative to $\vec{r}$ and $\vec{r}\,'$)

$$\int Q(\vec{r},t)\,\frac{\partial^2}{\partial t^2}\,Q(\vec{r}\,',t)\,d\vec{r}\,d\vec{r}\,'$$

$$= \int \left\{\frac{\partial}{\partial t}\left(Q(\vec{r},t)\,\frac{\partial}{\partial t}\,Q(\vec{r}\,',t)\right) - \frac{\partial Q(\vec{r},t)}{\partial t}\,\frac{\partial Q(\vec{r}\,',t)}{\partial t}\right\} d\vec{r}\,d\vec{r}\,'$$

$$= \frac{d}{dt}\int Q(\vec{r},t)\,\frac{\partial}{\partial t}\,Q(\vec{r}\,',t)\,d\vec{r}\,d\vec{r}\,' - \left(\frac{d}{dt}\int Q(\vec{r},t)d\vec{r}\right)^2\ , \qquad (77)$$

and

$$\int |\vec{r}-\vec{r}\,'|\,Q(\vec{r},t)\,\frac{\partial^3}{\partial t^3}\,Q(\vec{r}\,',t)\,d\vec{r}\,d\vec{r}\,'$$

$$= \int |\vec{r}-\vec{r}\,'|\left\{\frac{\partial}{\partial t}\left(Q(\vec{r},t)\,\frac{\partial^2}{\partial t^2}\,Q(\vec{r}\,',t) - \frac{\partial Q(\vec{r},t)}{\partial t}\,\frac{\partial Q(\vec{r}\,',t)}{\partial t} + \frac{\partial^2 Q(\vec{r},t)}{\partial t^2}\,Q(\vec{r}\,',t)\right)\right\} d\vec{r}\,d\vec{r}\,'$$

$$- \int |\vec{r}-\vec{r}\,'|\,\frac{\partial^3 Q(\vec{r},t)}{\partial t^3}\,Q(\vec{r}\,',t)\,d\vec{r}\,d\vec{r}\,'\ ,$$

or (on the basis of the aforementioned symmetry)

$$\int |\vec{r}-\vec{r}\,'|Q(\vec{r},t)\ \frac{\partial^3 Q(\vec{r}\,',t)}{\partial t^3}\ d\vec{r}\ d\vec{r}\,'$$
$$= \frac{d}{dt} \int\ |\vec{r}-\vec{r}\,'|\left\{Q(\vec{r},t)\ \frac{\partial^2}{\partial t^2}\ Q(\vec{r}\,',t)\ -\ \frac{1}{2}\ \frac{\partial Q(\vec{r},t)}{\partial t}\ \frac{\partial Q(\vec{r}\,',t)}{\partial t}\right\} d\vec{r}\ d\vec{r}\,' \tag{78}$$

and, finally,

$$\int |\vec{r}-\vec{r}\,'|^2 Q(\vec{r},t)\ \frac{\partial^4 Q(\vec{r}\,',t)}{\partial t^4}\ d\vec{r}\ d\vec{r}\,'$$
$$= \frac{d}{dt} \int\ |\vec{r}-\vec{r}\,'|^2\left\{Q(\vec{r},t)\ \frac{\partial^3 Q(\vec{r}\,',t)}{\partial t^3}\ d\vec{r}\ d\vec{r}\,'\ -\ \frac{\partial Q(\vec{r},t)}{\partial t}\ \frac{\partial^2 Q(\vec{r}\,',t)}{\partial t^2}\right\} d\vec{r}\ d\vec{r}\,'$$
$$+ \int\ |\vec{r}-\vec{r}\,'|^2\ \frac{\partial^2 Q(\vec{r},t)}{\partial t^2}\ \frac{\partial^2 Q(\vec{r}\,',t)}{\partial t^2}\ d\vec{r}\ d\vec{r}\,'\ , \tag{79}$$

it follows that

$$I = \frac{d\varepsilon}{dt} + P$$

where

$$P = \frac{1}{4\pi\rho_0 c}\left(\frac{d}{dt} \int\ Q(\vec{r},t)\ d\vec{r}\right)^2 - \frac{1}{24\pi\rho_0 c^3} \int\ |\vec{r}-\vec{r}\,'|^2\ \frac{\partial^2 Q(\vec{r},t)}{\partial t^2}\ \frac{\partial^2 Q(\vec{r}\,',t)}{\partial t^2}\ d\vec{r}\ d\vec{r}\,'$$
$$+\ 0(1/c^5) \tag{80}$$

and

$$\varepsilon = \frac{1}{8\pi\rho_0} \int \frac{Q(\vec{r},t)Q(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|}\ d\vec{r}\ d\vec{r}\,'\ -\ \frac{1}{8\pi\rho_0 c}\ \frac{d}{dt}\left(\int\ Q(\vec{r},t)d\vec{r}\right)^2 + 0\left(\frac{1}{c^2}\right)\ . \tag{81}$$

The prior representations (29) and (35), appropriate to the case of a point monopole or dipole, are directly recovered from the first and second terms of (80), respectively, on deploying the source functions (24) and (32).

If a distributed though unidirectional force $\vec{F}(\vec{r},t) = \vec{i}\mathscr{F}(\vec{r},t)$ acts the power characterization analogous to (80) has the form

$$P = \frac{1}{4\pi\rho_0 c^3}\left(\int\ \frac{\partial\mathscr{F}(\vec{r},t)}{\partial t}\ d\vec{r}\right)^2$$
$$-\ \frac{1}{120\pi\rho_0 c^5} \int\ |\vec{r}-\vec{r}\,'|^2\ \frac{\partial^2\mathscr{F}(\vec{r},t)}{\partial t^2}\ \frac{\partial^2\mathscr{F}(\vec{r}\,',t)}{\partial t^2}\ d\vec{r}\ d\vec{r}\,' + \ldots\ . \tag{82}$$

## References

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